ON THE DIOPHANTINE EQUATION OF SECOND DEGREE OF THE FORM

$3XY = Z(X+Y)$

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Abstract:

The ternary quadratic Diophantine equation given by $3XY = Z(X+Y)$ is analyzed for its non-zero distinct integer points on it. A few interesting relations between the solutions and special number patterns namely Polygonal number, Pyramidal number, Pronic number, Stella Octangular number, Octahedral number and Nasty numbers are presented.

Keywords: Ternary, quadratic, integral solutions, Diophantine equation, unit fraction, Egyptian fraction.

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INTRODUCTION:

The ternary quadratic Diophantine equations offer an unlimited field for research due to their variety [1,25]. For an extensive review of various problems, one may refer [2-3,4-20,23,24]. In [21], a few integer solutions of the Diophantine equation $3XY = n(X+Y)$, have been obtained by employing the Egyptian fraction representation of the number $\frac{3}{n}$. In this context one may refer[4,22]. These results have motivated us for obtaining infinitely many non-zero distinct positive integer solutions to the ternary quadratic Diophantine equation given by $3XY = Z(X+Y)$. Also, a few interesting relations among the solutions are presented.

NOTATIONS:

- P_n^m : Pyramidal number of rank n with m sides
- $T_{m,n}$: Polygonal number of rank with m sides
- Pr_n : Pronic number of rank n
- SO_n : stella Octangular number of rank n
- OH_n : Octhedral number of rank n

METHOD OF ANALYSIS:

The ternary quadratic Diophantine equation to be solved for its non-zero integer solutions is given by

$$3XY = Z(X+Y) \quad (1)$$

To start with, it is noted that (1) is satisfied by

$(X, Y, Z) = (2,2,3), (k+1, k[k+1], 3k), (2,1,2), (3-k, k, k[3-k])$

However, we have an another choice of solutions which is illustrated below.

The substitution of the linear transformations

$$X = u+v, Y = u-v, (u \neq v \neq 0) \quad (2)$$

in (1) leads to $3u^2 - 2uz - 3v^2 = 0$ (3)

Treating (3) as a quadratic in u and solving for u, we get

$$u = \frac{1}{3}[Z \pm \sqrt{Z^2 + 9v^2}] \quad (4)$$

Employing the standard solutions of the Pythagorean equation, the square-root on R.H.S of (4) is eliminated when

$$Z = 2mn, v = \frac{1}{3}(m^2 - n^2), (m > n > 0) \quad (5)$$

As our interest is on finding integer solutions, note that v is an integer

when m and n are replaced by 3M and 3N (M>N) respectively.

Then, we have

$$Z = 18MN, v = 3(M^2 - N^2), \ M > N > 0 \quad (6)$$

and from (4), $u = 6MN \pm 3(M^2 + N^2)$

Substituting the above values of u and v in (2), the two sets of non-zero distinct integer solutions to (1) are as follows:

SET 1:

$$X(M, N) = 6M(M + N)$$
$$Y(M, N) = 6N(M + N)$$
$$Z(M, N) = 18MN$$

PROPERTIES:

- $X(1, N) + Y(1, N) - T_{14,N} \equiv 6 \pmod{17}$
- $X(N^2, N+1) + Y(N^2, N+1) - 24P_N^5 - T_{26,N} \equiv 6 \pmod{23}$
- $Y(M, 2M^2 - 1) + Z(M, 2M^2 - 1) - 24SO_M + 24T_{4,M^2} - 24T_{4,M} \equiv 0 \pmod{6}$
- $X(M,1) + Z(M,1) - 6T_{4,M} \equiv 0 \pmod{3}$
- $X(M,1) + Y(M,1) - T_{10,M} - T_{6,M} \equiv 0 \pmod{2}$
- $X(2, N) + Y(2, N) - T_{14,N} \equiv N \pmod{4}$

- $Y(M+1[M+2],M)-6T_{4,M}=36P_M^3$
- $Y(M,M+1)-6T_{4,M}=6\,Pr_M$
- $Z(N,N)-Y(N,N)$ is a nasty number
- $X(M,M)+Y(M,M)$ is a nasty number

SET 2:

$$X(M,N)=6N(M-N)$$
$$Y(M,N)=-6M(M-N)$$
$$Z(M,N)=18MN$$

PROPERTIES:

- $X(M,1)-Y(M,1)-6T_{4,M}\equiv 0(\bmod 3)$
- $X(M,1)+Y(M,1)+T_{14,M}\equiv -6(\bmod 7)$
- $Y(N^2,N+1)-12P_N^5+T_{12,N}+T_{4,N}\equiv 0(\bmod 2)$
- $X(M,1)+Z(M,1)\equiv 0(\bmod 6)$
- $Y(1,2N)\equiv 0(\bmod 3)$
- $Y(N,2N^2+1)+6T_{4,N}=18OH_N$
- $X(N,N+1[N+2])+6T_{4,N}=36P_N^3$
- $X(N+1,N)+6T_{4,N}=6\,Pr_N$
- $X(-1,N)+6\,Pr_N=0$
- $2\{X(M,M)+Y(M,M)+Z(M,M)\}$ is a nasty number

REMARK:

It is worth mentioning here that, the square-root on the R.H.S of (4) is also eliminated when

$$Z=m^2-n^2,v=\frac{2}{3}mn,m>n>0.$$

Employing the procedure similar to the above, one obtains two sets of non-zero distinct integer solutions to (1) which are presented below:

SET 3:

$$X(m,n) = 6m^2 + 2mn$$
$$Y(m,n) = 6m^2 - 2mn$$
$$Z(m,n) = 9m^2 - n^2$$

SET 4:

$$X(m,n) = 6n(m-n)$$
$$Y(m,n) = -6n(m+n)$$
$$Z(m,n) = 9(m^2 - n^2)$$

CONCLUSION:

In this paper we have presented infinitely many non-zero integer solutions to the ternary quadratic Diophantine equation $3XY = Z(X+Y)$. It seems that the positive integer solutions presented in this paper are different from those in [21].

REFERENCES:

[1] Dickson, L.E., History of Theory of Numbers, Vol.2,Chelsea Publishing company, . NewYork, 1952

[2] Divya,S., Gopalan, M.A., Vidhyalakshmi, S., Lattice points on the homogeneous cone $X^2 + Y^2 = 40Z^2$,the Experiment, vol.17(3), 1191-1199, 2013

[3] Meena, K., Vidhyalakshmi, S., Divya, S., Gopalan, M.A., Integral points on the cone $Z^2 = 41X^2 + Y^2$,Scholars journal of engineering and technology, sch.j.eng.tech, 2(2B), 301-304, 2014

[4] Erdos, P., Straus, E.G., On a Diophantine equation, Math. Lapok, 1:192-210, 1950.

[5] Gopalan, M.A.,Pandichevi, V., Integral solution of ternary quadratic equation $z(x+y) = 4xy$, Actociencia Indica, ,Vol. XXXIVM, No.3, 1353-1358, 2008

[6] Gopalan, M.A.,Kalinga Rani, J., Observation on the Diophantine equation, $y^2 = Dx^2 + z^2$

Impact J.sci tech ; Vol (2), 91-95, 2008

[7] Gopalan, M.A.,Pandichevi, V., on ternary quadratic equation $x^2 + y^2 = z^2 + 1$, Impact J.sci tech; ,Vol 2(2), 55-58. 2008

[8] Gopalan, M.A., Manju somanath, Vanitha,N., Integral solutions of ternary quadratic Diophantine equation $x^2 + y^2 = (k^2 + 1)^n z^2$. Impact J.sci tech; Vol 2(4), 175-178, 2008,

[9] Gopalan, M.A., Manju somanath, Integral solution of ternary quadratic Diophantine equation $xy + yz = zx$ AntarticaJ,Math, 1-5,5(1), 2008.

[10] Gopalan,M.A.,and Gnanam,A., Pythagorean triangles and special polygonal numbers, international Journal of Mathematical Science, Vol.(9),No.1-2,211-215, Jan-Jun 2010

[11] Gopalan, M.A., and Vijayasankar, A.,Observations on a Pythagorean problem, Acta Ciencia Indica , Vol.XXXVIM, No.4,517-520, 2010,

[12] Gopalan.M.A., and pandichelvi.V., Integral solutions of ternary quadratic equation $z(x - y) = 4xy$, Impact J.sci TSech; Vol (5),No.1,01-06, 2011

[13] Gopalan, M.A.,Kalinga Rani, J.On ternary quadratic equation $x^2 + y^2 = z^2 + 8$,Impact J.sci tech ; Vol (5), no.1,39-43,2011

[14] Gopalan, M.A., Geetha, D., Lattice points on the hyperbolid of two sheets $x^2 - 6xy + y^2 + 6x - 2y + 5 = z^2 + 4$, Impact J.sci tech ; Vol(4),No.1,23-32,2010

[15] Gopalan, M.A., Vidhyalakshmi, S., and Kavitha, A., Integral points on the homogeneous Cone $z^2 = 2x^2 - 7y^2$, DiophantusJ.Math., 1(2),127-136, 2012

[16] Gopalan, M.A., Vidhyalakshmi, S., Sumathi,G., Lattice points on the hyperboloid one sheet $4z^2 = 2x^2 + 3y^2 - 4$, DiophantusJ.math., 1(2),109-115,2012.

[17] Gopalan, M.A., Vidhyalakshmi, S., and Lakshmi,K., Integral points on the hyperboloid of two sheets $3y^2 = 7x^2 - z^2 + 21$, DiophantusJ.math., 1(2),99-107,2012

[18] Gopalan, M.A., and Srividhya,G., Observations on $y^2 = 2x^2 + z^2$ Archimedes J.Math,

2(1), 7-15, 2012

[19] Gopalan,M.A., Sangeetha,G.,Observation on $y^2 = 3x^2 - 2z^2$ AntarcticaJ.Math, 9(4), 359-362, 2012

[20] Gopalan,M.A., and Vijayalakshmi,R., On the ternary quadratic equation $x^2 = (\alpha^2 - 1)(y^2 - z^2)$, α>1, Bessel J.Math, 2(2),147-151, 2012

[21] Hari kishan., Megha Rani., and Smiti Agarwal., The Diophantine equations of second and Higher degree of the form $3xy = n(x+y)$ and $3xyz = n(xy+yz+zx)$ etc, Asian journal of Algebra, 4(1), 31-37, 2011

[22] Jaroma, J.H., On expanding $\frac{4}{n}$ into three Egyptian fractions, Crux Math, 30: 36-37, 2004

[23]Manjusomanath, Sangeetha,G., Gopalan,M.A., On the homogeneous ternary quadratic Diophantine equation $x^2 + (2k+1)y^2 = (k+1)^2 z^2$, Bessel J.Math, 2(2),107-110, 2012

[24] Manjusomanath, Sangeetha,G., Gopalan,M.A., Observations on the ternary quadratic equation $y^2 = 3x^2 + z^2$, Bessel J.Math, 2(2),101-105, 2012

[25] Mordell, L.J., Diophantine equations, Academic press, New York, 1969